Relativistic Doppler Effect and Group Velocity in Homogeneous Nonlinear and Inhomogeneous Plasmas

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We study the variations of the received frequencies (Complex Doppler effect) from a source moving uniformly in a homogeneous isotropic plasma and emitting a uniform EM plane wave with amplitude E_0, as a function of ω_e/E_0 where ω_e is the plasma frequency. The ratio of the velocity U of the source over the group velocity U_{gr} of the transmitted wave is then calculated from the slope of the Doppler curves.

When the plasma is inhomogeneous (one-dimensional stratification) the Doppler effect depends on the particular profile of the refractive index and its higher (even) derivatives with respect to the direction of stratification z. In this case the ratio $U/U_{gr}=f(z)$ is given as the solution of a higher order differential equation with inhomogeneous part depending on the slope of the Doppler curve with respect to frequency. Typical cases involving symmetrical and transition profiles are examined. Harmonic generation in the plasma is not taken into account.

A. Doppler Shift in Nonlinear Homogeneous Media

The relativistic Doppler effect in a homogeneous isotropic plasma has been studied so far [1] under the assumption of zero-field-intensity for the EM wave transmitted from the moving source. This means that in the evaluation of the refractive index of the medium the Electromagnetic pressure has been altogether omitted vs. the thermal pressure of the gas or, the plasma has been considered at zero temperature [refractive index $n(\omega)=\{1-\omega_e^2/\omega^2\}^{1/2}$]. In any realistic case however, involving the motion of EM sources in dispersive media it is obvious that the intensity of the transmitted waves should be taken into account. It has been proved [2] that such a biasing Electric field $\bar{E}_0=\bar{E}(\bar{r})^{-j\omega t}$ modifies the cold-plasma refractive index as following:

$$n(\omega)=\{1-(\omega_e^2/\omega^2)\,e^{-E^2/\omega^2}\}^{1/2} \tag{1}$$

where $E=e\,|\bar{E}_0|/\sqrt{8\,m\,k\,T}$, e, m are the electron charge and mass, k is the Boltzmann's constant and T is the plasma temperature. Eventually Eq. (1) describes a non-homogeneous medium and for such a medium the principle of phase invariance from which the Doppler shift is deduced [1], is not valid. However, far from the source where the field can be considered quasi-plane, E may be taken as constant: consequently $n(\omega)$ in Eq. (1) is taken as an $\bar{r}$-independent function.

In Figs. 1 we plot the ratio of the received frequency ω over the plasma frequency as a function of the transmitted frequency ω_0 in the rest frame of the source vs. ω_e for given values of ω_e/E. The relation between ω_0 and ω is

$$\omega_0=\frac{\omega}{\sqrt{1-\beta^2}}\,[1-\beta\,n(\omega)] \tag{2}$$

with $\beta=U/c$ and for motion of the source parallel to the direction of the wave number of the transmitted wave. Expression (2) above is deduced by applying the phase invariance method (under Lorentz transformations) to the phase of the traveling plane wave. E is evaluated at the observer's frame.

In the case $E\rightarrow 0$ it is known that for a receding source ($\beta<0$) one frequency ω is received for a given ω_0 and for an approaching source ($\beta>0$) two frequencies ω are received for any ω_0 between $\omega_e/\sqrt{1-\beta^2}$ and ω_e. As the sequence of Figs. 1 shows for $E\neq 0$, new branches of the Doppler effect emerge from the origin — both for $\beta>0$ and $\beta<0$. These new branches are separated from the "old" ones by cut-off regions. The two branches for each β approach as ω_e/E decreases, and for $\omega_e/E \sim 1.65$ they merge. As E/ω_e increases further we have again one branch — starting now from the origin — for each β and as $E/\omega_e\rightarrow\infty$, $n(\omega)\rightarrow 1$ (the plasma is squeezed away from the source to infinity) and one gets $\omega_0\rightarrow\omega\cdot\sqrt{(1-\beta)/(1+\beta)}$, i.e. the vacuum case.

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[1] CHARLES H. PAPAS, Theory of Electromagnetic Propagation, McGraw-Hill, New York 1965, Chapt. 7.

[2] A. V. GUREVICH and L. P. PITAEVSKII, Sov. Phys. JETP **5**, (18 No. 3), 855 [1964].

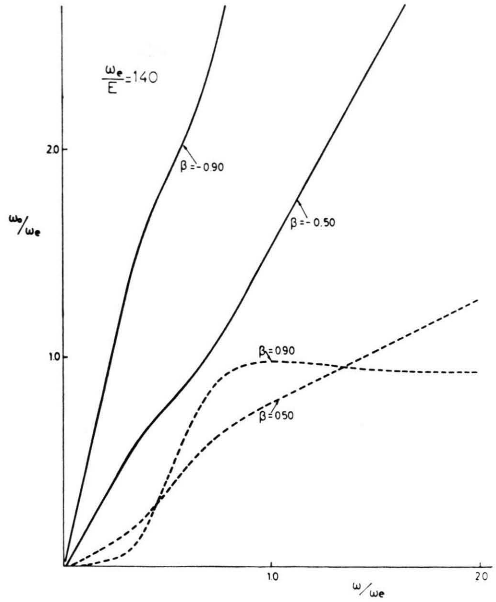

Fig. 1.1.

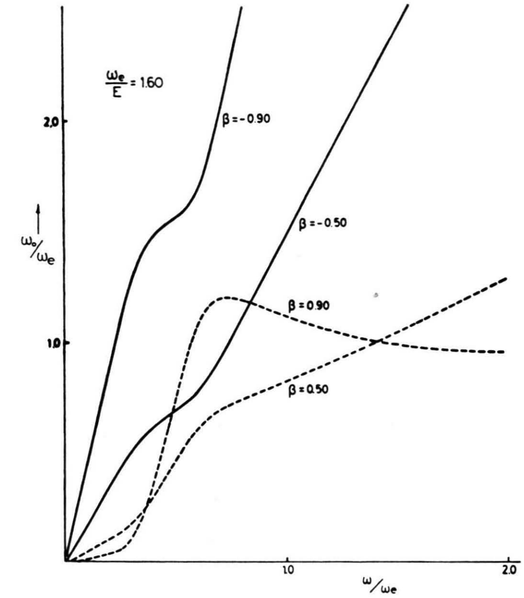

Fig. 1.2.

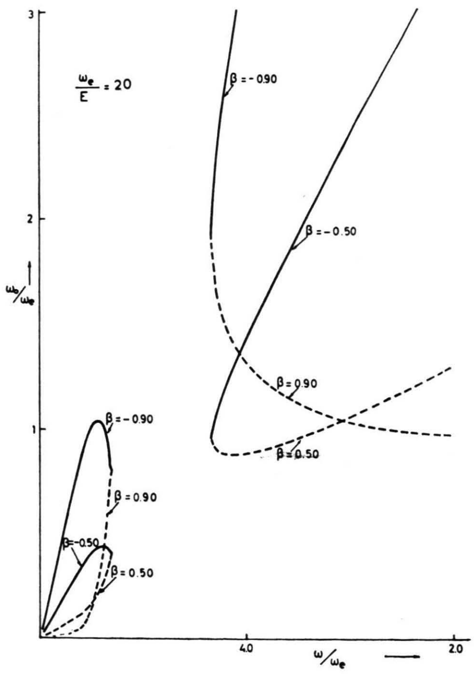

Fig. 1.5.

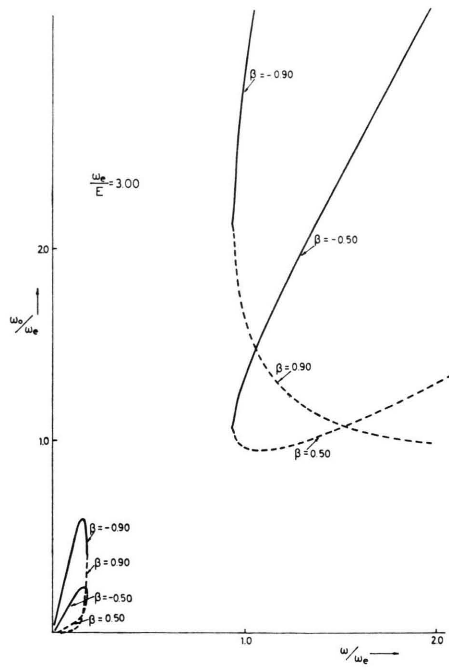

Fig. 1.6.

The interesting feature with the above curves is that now for a given value of ω_0 there correspond two values of ω to any β positive *or* negative and that near the critical region of coalition we receive *three* distinct frequencies.

From (2) we get

$$\frac{d\omega_0}{d\omega} = \frac{1}{\sqrt{1-\beta^2}}\,[1-\beta\, n(\omega)] - \frac{\beta\,\omega}{\sqrt{1-\beta^2}}\,\frac{dn(\omega)}{d\omega} = \\ = \frac{1}{\sqrt{1-\beta^2}}\left(1-\frac{U}{U_{gr}}\right) \qquad (3)$$

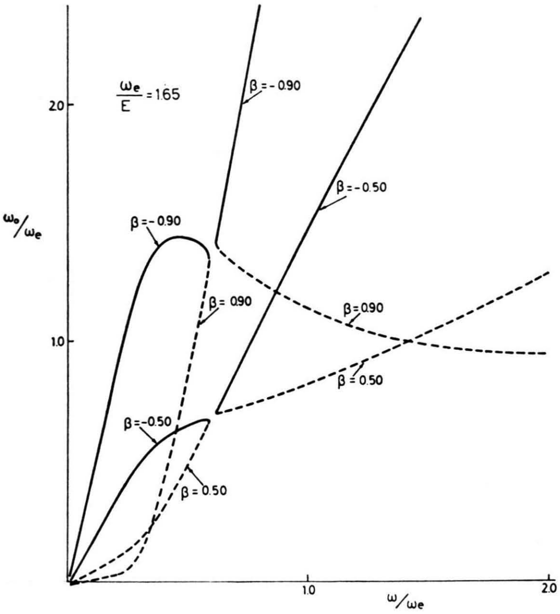

Fig. 1.3.

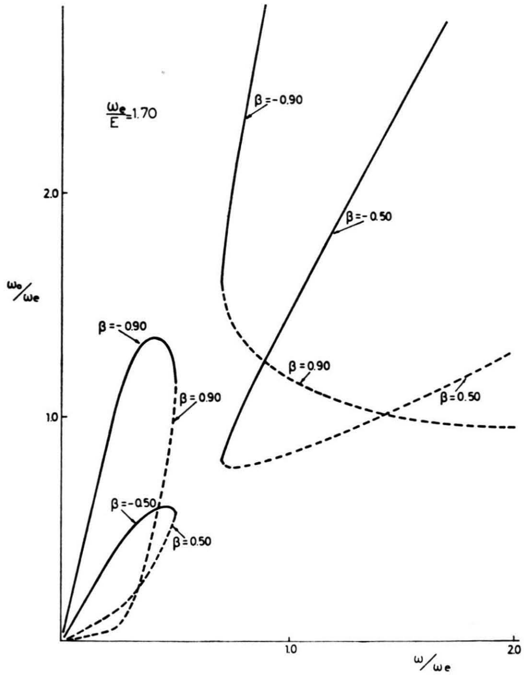

Fig. 1.4.

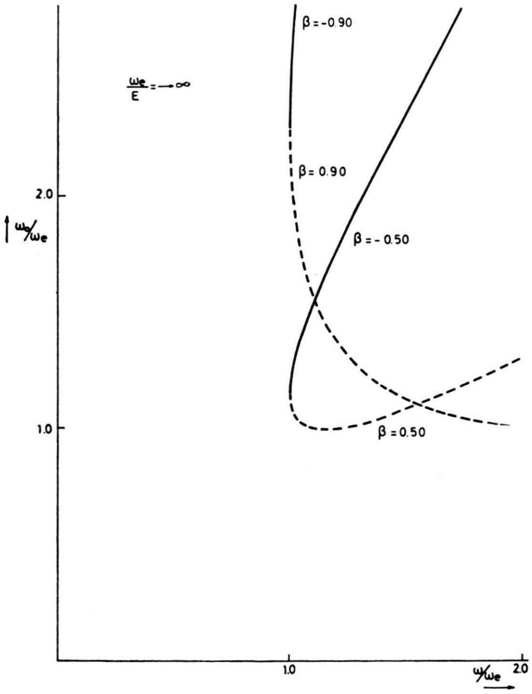

Fig. 1.7.

since

$$\frac{c}{U_{gr}} = n(\omega) + \omega \frac{\partial n(\omega)}{\partial \omega}\,.$$

Here, it has been implicitly assumed that the group velocity in a non-linear medium is given by the same expression as in the usual case of a linear medium, i. e. $U_{gr} = \partial\omega(K, E^2)/\partial K \cong \partial\omega(K, 0)/\partial K$. In general, for highly non-linear media, i. e. for strong fields E this assumption is *not* correct since from the dispersion relation $\omega = f(K, E^2)$ one gets:

$$\omega \cong \omega_0(K) + \left(\frac{\partial\omega}{\partial E^2}\right)_0 E^2 + \text{(higher-order terms)}$$

where $\omega_0(K)$ determines the dispersion law in the linear approximation. (For a lucid discussion see Ref. [6].) However, in our case, the variable is the ratio ω_e/E (from $0 \ldots \infty$) and small values of this variable do not necessarily imply large values for the normalized field E (see note added in proof at the end of the text).

So, the ratio $Y = U/U_{gr}$ can be evaluated from the curves of Figs. 1. Direct calculation of the group velocity

$$U_{gr} \cong \frac{\partial\omega}{\partial K} = \frac{c\sqrt{1-(\omega_e^2/\omega^2)\, e^{-E^2/\omega^2}}}{1-(\omega_e^2/\omega^2)\,(E^2/\omega^2)\, e^{-E^2/\omega^2}}$$

shows that, due to the very steep variation of $n(\omega)$ with ω, the above expression does not represent the velocity of energy transport beyond a certain limit for which U_{gr} exceeds c. This happens because in this case higher order derivatives of ω with respect

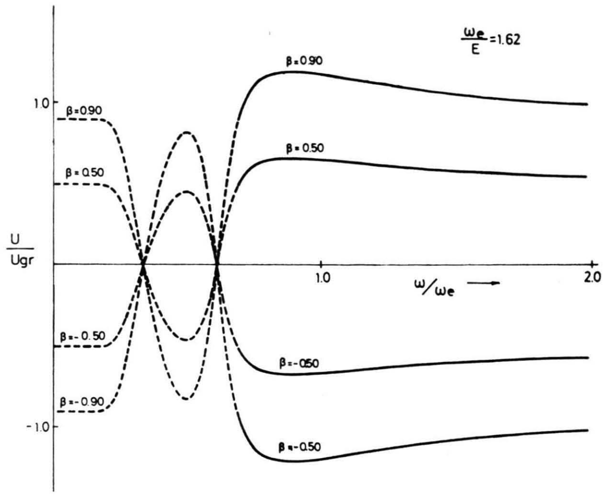

Fig. 2.1.

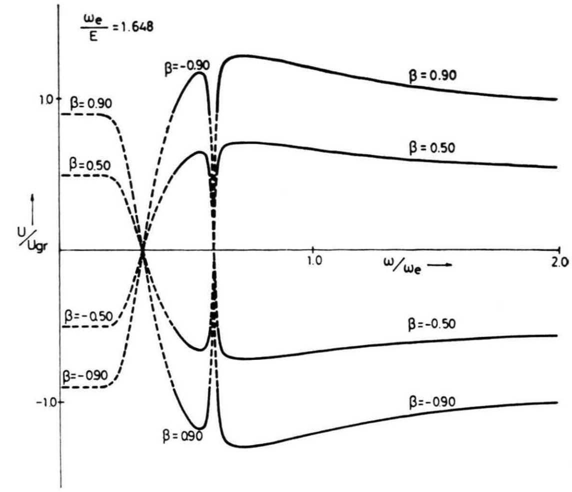

Fig. 2.2.

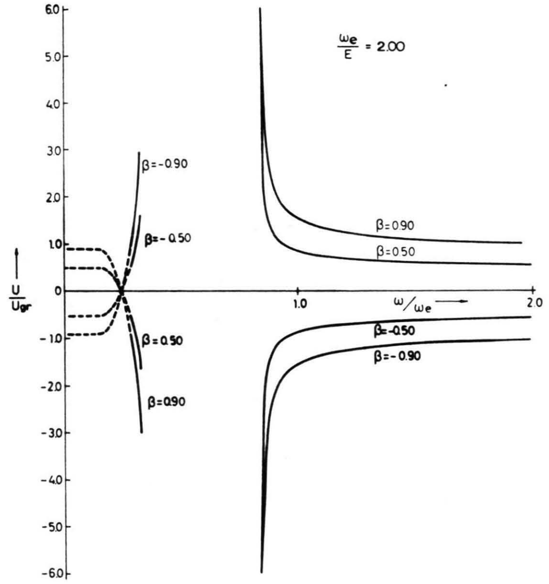

Fig. 2.5.

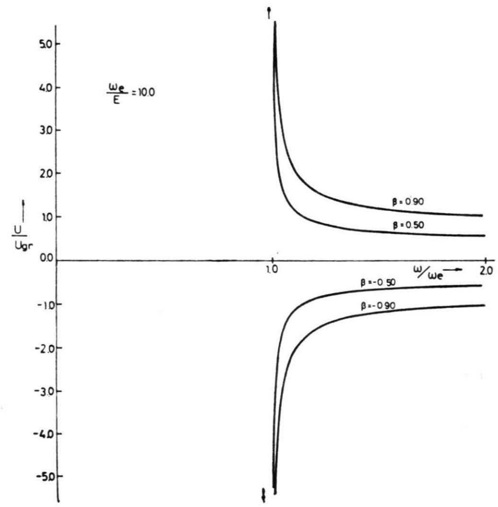

Fig. 2.6.

to the wavenumber K cannot be omitted. A first-order correction intending to take into account a few higher derivatives $\partial^2\omega/\partial K^2$, $\partial^3\omega/\partial K^3$ (l. c. [4]) gives for the present case:

$$U_{gr} \cong \frac{24\, c \sqrt{1-(\omega_e^2/\omega^2)\, e^{-E^2/\omega^2}}}{24\left\{1-\dfrac{\omega_e^2}{\omega^2}\dfrac{E^2}{\omega^2}\, e^{-E^2/\omega^2}\right\} + \tau^2\, \omega_e^2\, e^{-E^2/\omega^2}\left\{3-\dfrac{27\,E^2}{\omega^2}+24\,\dfrac{E^4}{\omega^4}-4\,\dfrac{E^6}{\omega^6}\right\}}$$

where $\tau \cong 10^{-23}$ sec is the natural line width constant. This correction prevents U_{gr} from acquiring infinite values, but otherwise gives poor results for a particularly steep profile $n(\omega)$ such as the one considered here.

It appears that in such cases a "velocity of energy-transport" cannot be defined unambiguously. In Figs. 2 the plots of U/U_{gr} deduced from Eq. (3) are displayed in full line in the allowed regions, where U_{gr} can be defined in the conventional way

[4] J. Nicolis, Z. Naturforsch. **26 a**, 124 [1971].

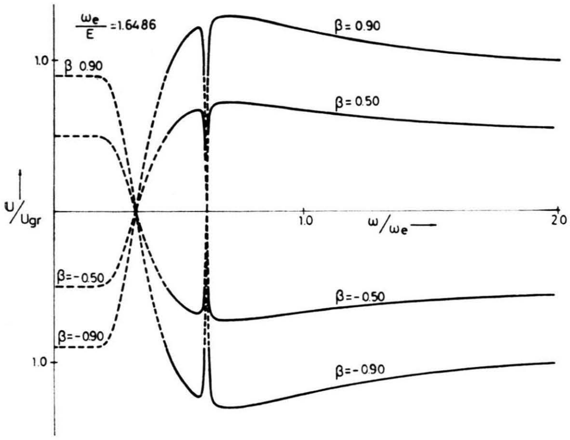

Fig. 2.3.

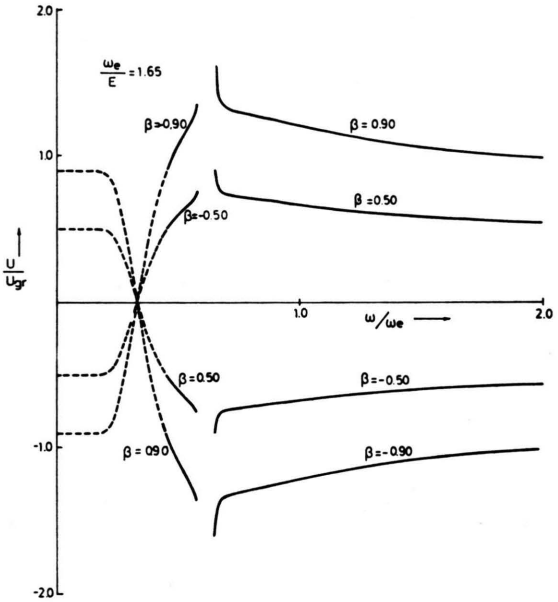

Fig. 2.4.

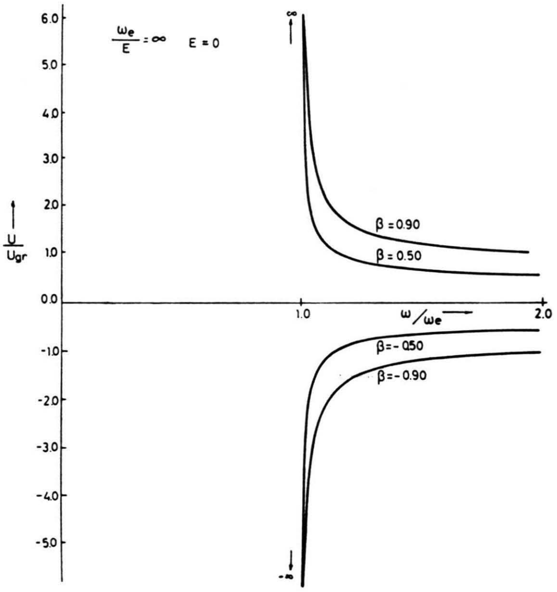

Fig. 2.7.

i. e. for $|U/U_{gr}| > |\beta|$. For ω_e/E below the critical value ~ 1.65, U_{gr} undergoes a single minimum at frequencies $\omega < \omega_e$. As ω_e/E increases beyond this critical value, cut-off regions appear and the shape of the curves changes: There appears now a peak of U/U_{gr} at frequencies $\omega < \omega_e$ approaching ω_e

and finally for $E \to 0$,

$$\frac{d(U/U_{gr})}{d\omega} = \frac{-\beta(\omega_e^2/\omega^3)}{[1-(\omega_e^2/\omega^2)]^{3/2}}$$

i. e. $U/U_{gr} \to \infty$ or $U_{gr} \to 0$, for $\omega \to \omega_e$.

For all cases $U/U_{gr} \to \beta$ as $\omega \to \infty$. It is now of some interest to evaluate the group velocity of the transmitted wave in the rest frame of the source, i. e. as seen from an observer moving with the source. Denoting by primes the parameters in the rest frame of the medium we write for the wave number and the frequency, the transformation relations [1]: (4)

$$\bar{K}' = \bar{K} - \gamma\frac{\omega}{c^2}\bar{U} + \frac{\bar{K}\cdot\bar{U}}{U^2}\bar{U}(\gamma-1), \quad \gamma = \frac{1}{\sqrt{1-\beta^2}},$$

$$\omega' = \gamma(\omega - \bar{K}\cdot\bar{U}) \qquad (5)$$

and $$\bar{U}_{gr} = \frac{\partial\omega}{\partial K} = \nabla_K(\omega) \text{ i. e. } U_{gr_{x,y}} = \frac{\partial\omega}{\partial K_{x,y}}. \qquad (6)$$

We assume $\bar{K}$ in the x, y plane, forming an angle Θ with the X direction; $\bar{U}$ is taken along the X axis. From Eqs. (4), (5) we get:

$$K_x' = K_x - \gamma\frac{\omega}{c}\frac{U}{c} + K_x(\gamma-1) = \gamma[K_x - \beta(\omega/c)],$$

$$K_y' = K_y, \quad K_z' = K_z,$$

and $$\omega' = \gamma(\omega - K_x U).$$

So, $$U'_{gr_x} = \frac{\partial\omega'}{\partial K_x'} = \frac{1}{(\partial K_x'/\partial\omega)(d\omega/d\omega')} = \frac{d\omega'/d\omega}{\partial K_x'/\partial\omega},$$

but $d\omega'/d\omega = \gamma(1 - U/U_{grx})$,

$$\partial K_x'/\partial\omega = \gamma(1/U_{grx} - \beta/c) \text{ so,}$$

$$U'_{gr_x} = \frac{U_{grx} - U}{1 - (U\,U_{grx}/c^2)} \text{ and } U_{gr_x} \equiv \frac{U'_{gr_x} + U}{1 + (U\,U'_{grx}/c^2)} \quad (7)$$

i. e. we get the usual expression expected from the Lorentz transformations of the velocity.

B. Doppler Shift in Inhomogeneous Media

In a (mildly) non-homogeneous medium (one-dimensional stratification), the Doppler shift can be calculated in an explicit way only for $U \ll c$. For motion parallel to the direction of stratification one obtains [3]

$$|\Delta\omega| \cong U \sum_{m=0}^{\infty} (-1)^m \frac{1}{(2K)^{2m}} \frac{d^{2m} K(z)}{dz^{2m}} \quad (10)$$

where $|\Delta\omega| = |\omega_0' - \omega|$ and $K(z)$ is the wavenumber as a function of "height". In cases where the series (10) converges, $K \gg 1$ is a desirable prerequisite for fast convergence.

There exist two obvious cases where (10) can be evaluated exactly: a) the sinusodial and the b) exponential profile.

a) For a harmonic profile we write

$$K = \frac{\omega}{c} \sqrt{\varepsilon_r} = A + \sin \varrho z > 1$$

where A is a constant and $\varrho = f(\omega)$.

Then,

$$\frac{d^{2m} K}{dz^{2m}} = (-1)^m \varrho^{2m} \sin z \quad \text{so,} \quad \frac{|\Delta\omega|}{U} = \sin(\varrho z) \sum_{m=0}^{\infty} (-1)^{2m} \left\{ \frac{\varrho}{2[A + \sin(\varrho z)]} \right\}^{2m} = \frac{\sin(\varrho z)}{1 - \lambda^2}$$

where

$$\lambda = \frac{\varrho}{2[A + \sin(\varrho z)]} = \frac{\varrho}{2K} \text{ under the condition } |\lambda| < 1.$$

$$\text{Now} \frac{\partial|\Delta\omega|}{\partial\omega} = U \left\{ \frac{\frac{\partial K}{\partial\omega}(1-\lambda^2) - (K-A)\, 2\lambda \frac{d\lambda}{d\omega}}{(1-\lambda^2)^2} \right\} =$$

$$= \frac{\frac{U}{U_{gr}}(1-\lambda^2) - 2\lambda(K-A) \frac{d\lambda}{d\omega} U}{(1-\lambda^2)^2} =$$

$$= \frac{Y(1-\lambda^2) + \lambda(K-A) \frac{\varrho(\partial K/\partial\omega) - K(\partial\varrho/\partial\omega)}{K^2} U}{(1-\lambda^2)^2}$$

and since

$$\frac{\partial K}{\partial\omega} = \varrho \cos(\varrho z) \frac{\partial\varrho}{\partial\omega}, \quad \frac{\partial\varrho}{\partial\omega} = \frac{\partial K}{\partial\omega} \frac{1}{\varrho \cos(\varrho z)}$$

we get finally

$$\frac{\partial|\Delta\omega|}{\partial\omega} = Y \frac{(1-\lambda^2) + 4(\lambda/\varrho)^3[\varrho^2 \cos(\varrho z) - K] \tan(\varrho z)}{(1-\lambda^2)^2} \quad (11)$$

from which Y is deduced as a function of the slope of the Doppler shift.

b) For an exponential profile, $K = (\omega/c)\, e^{-\gamma z}$, where γ is a constant so,

$$\partial^{2m} K/\partial z^{2m} = (-1)^{2m} \gamma^{2m} (\omega/c)\, e^{-\gamma z}$$

and

$$\frac{|\Delta\omega|}{U} = \frac{\omega}{c} e^{-\gamma z} \sum (-1)^m \frac{\gamma^{2m}}{(2K)^{2m}} = = K \sum (-1)^m \lambda^{2m} = \frac{K}{1+\lambda^2}$$

where $\lambda = \gamma/2K$, for $|\lambda| < 1$. Further,

$$\frac{\partial|\Delta\omega|}{\partial\omega} = \frac{Y}{1+\lambda^2} + U K \frac{\partial}{\partial\omega}\left(\frac{1}{1+\lambda^2}\right) = Y \frac{1+3\lambda^2}{(1+\lambda^2)^2}. \quad (12)$$

In the general case the relation (10) rarely converges when $|K(\omega, z)| < 1$. For a linear profile one gets $Y = \partial|\Delta\omega|/\partial\omega$.

Differentiation of both sides of (10) with respect to ω gives:

$$\frac{\partial|\Delta\omega|}{\partial\omega} = \sum_{m=0}^{\infty} (-1)^m \left\{ -2^{-2m+1} K^{-2m-1} m \frac{d^{2m} K}{dz^{2m}} Y + \left(\frac{1}{2K}\right)^{2m} \frac{d^{2m} Y}{dz^{2m}} \right\}$$

For $m = 1$ (quadratic model) one gets:

$$\frac{d^2 Y}{dz^2} - \left\{ 4K^2 + \frac{2}{K} \frac{d^2 K}{dz^2} \right\} Y = -4K^2 \frac{\partial|\Delta\omega|}{\partial\omega}$$

and for $K \sim a z^2$ the above becomes

$$\frac{d^2 Y}{dz^2} - \left\{ 4a^2 z^4 + \frac{4}{z^2} \right\} Y = -4a^2 z^4 \frac{\partial|\Delta\omega|}{\partial\omega} \quad (13)$$

In Figs. 3 a, b and 4 a, b we display $|\Delta\omega|/U$ vs. ω_e/ω and $|\Delta\omega|/U$ vs. z for the cases of a symmetric

$$\left(X = \frac{\omega_e^2}{\omega^2} = \frac{\omega^2_{oe}}{\omega^2} \left(\frac{1-e^z}{1+e^z}\right)^2 \right)$$

and a transition ionospheric layer

$$\left(X = \frac{\omega_e^2}{\omega^2} = \frac{\omega^2_{oe}}{\omega^2} \frac{1}{1+e^z} \right), \quad -2\pi < z < 2\pi,$$

in which in addition a small-scale stochastic variation of the electron distribution $\Delta N/N$ has been superimposed.

The expression for the "average" wave number for such an irregular slightly (cold) ionized gas has been developed recently [5] and for isotropic ir-

[3] K. S. H. Lee and Ch. H. Papas, J. Math. Phys. **42**, (3), 189 [1963].

[5] L. S. Taylor, J. Geophys. Research **67**, No. 10, 3843 [1962].

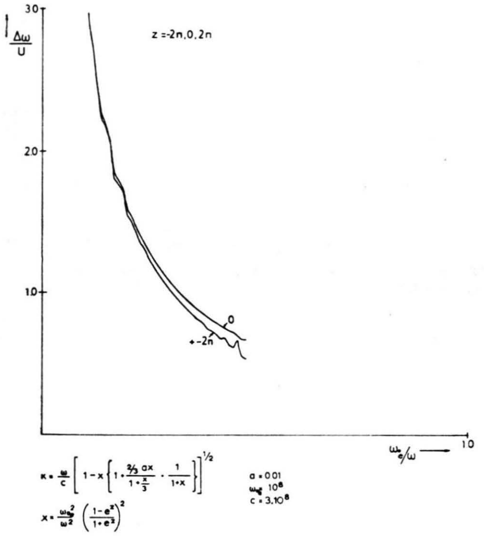

Fig. 3.1.

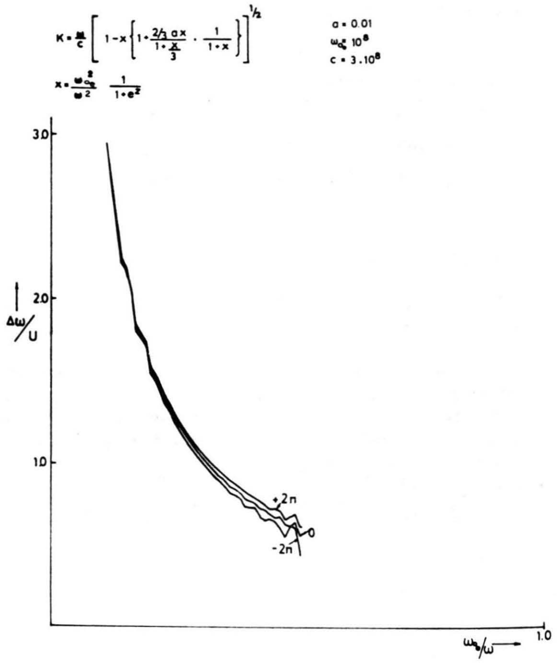

Fig. 3.2.

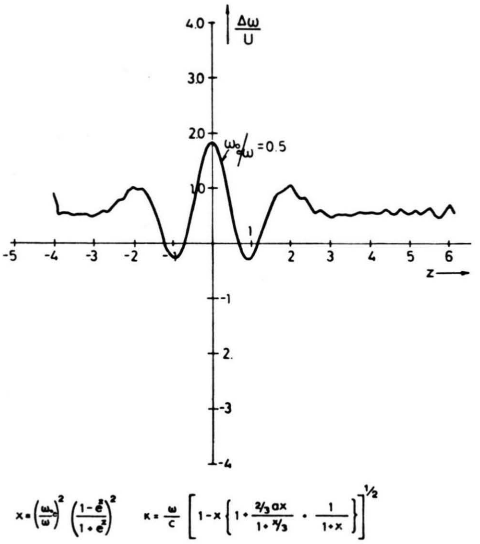

Fig. 4.1.

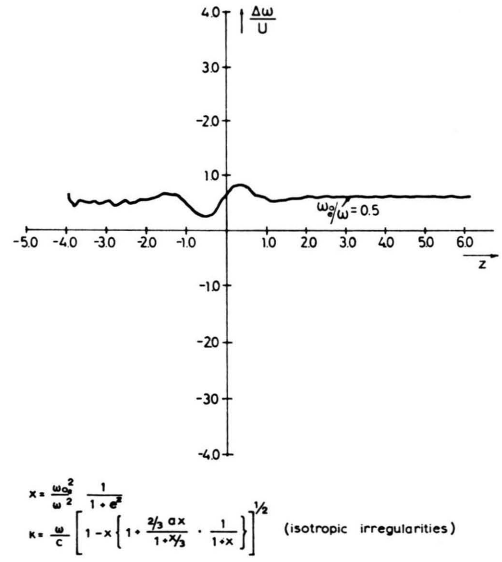

Fig. 4.2.

regularities it is given as:

$$K = \frac{\omega}{c}\left\{1 - X\left[1 + \frac{\frac{2}{3}\,a\,X}{1+\frac{1}{3}X}\cdot\frac{1}{1+X}\right]\right\}^{1/2}$$

where $a = \langle (\Delta N/N)^2 \rangle$.

In the present case we have taken $a = 0.01$ and $\omega_{0e} = 10^8$ Hz. The numerical calculations have been extended up to $m = 10$ (tenth derivative). In the study of the variation of $|\Delta\omega|/U$ with respect to ω_e/ω , beyond a certain limit ($\omega_e/\omega \sim 0.8$) $K(\omega, z)$ becomes <1 so that $1/(2K)^{2m} \gg 1$ and the series (10) beyond that limit oscillates rapidly even for small values of the corresponding derivatives with respect to z. We have plotted in Figs. 3 a, 4 a the corresponding curves within the frequency range for which the series converges rather rapidly ($K > 1$).

The variations of $|\Delta\omega|/U$ with respect to z are more pronounced near the middle of the layers for both the symmetric and the transitional case. For the symmetric layer the Doppler shift can become zero or even be slightly reversed *.

Acknowledgements

The authors thank Dr. E. K. YFANTIS for helpful discussions.

[6] M. J. LIGHTHILL, J. Inst. Math. Appl. **1**, 269 [1965]; Proc. Roy. Soc. London A **299**, 28 [1967].

* *Note added in proof:* G. B. WHITHAM (Proc. Roy. Soc. London A **283**, 238 [1965]; A **299**, 1 [1967]), has proved that in – one dimensional – non linear dispersive systems, wave-number (or amplitude) are propagated at *two* different velocities because the fundamental equations governing the propagation are hyperbolic: this means that non-linearity in fact "splits" the group velocity. — In the limiting case of very small amplitudes however, the equations are parabolic and only *one* velocity of propagation occurs. In our case Whitham's results should give:

$$U_{\text{gr}} = (\partial\omega/\partial K)_0 \pm E\sqrt{(\partial\omega/\partial E^2)_0\,(\partial^2\omega/\partial K^2)_0} + O(E^2)$$

where $\omega = f(K, E^2)$ is the pertinent non-linear dispersion relation. — In our problem however, the transmitted E.M. wave is excited from a fixed source i. e. a source *independent* of the medium. Consequently (since Maxwell's equations are linear!) the frequency of the E.M. waves is necessarily amplitude independent and the used formula for the group velocity in the text is rigorous. What is "split" here, therefore, is the group velocity of the plasma (longitudinal) waves which are excited by the strong electromagnetic field — a case of no concern here. — It would be of course otherwise if the transmitted E.M. wave is excited from the (non-linear) motion of the plasma particles themselves.